A SUPPLEMENT TO FUJINO'S PAPER: ON ISOLATED LOG CANONICAL SINGULARITIES WITH INDEX ONE

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Abstract. Let E be the essential part of the exceptional locus of a good resolution of an isolated, log canonical singularity of index one. We describe the dimension of the dual complex of E in terms of the Hodge type of $H^{n-1}(E, \mathcal{O}_E)$, which is one of the main results of the paper [1] of Fujino. Our proof uses only an elementary classical method, while Fujino's argument depends on the recent development in minimal model theory.

In this paper, a normal singularity (X, x) of dimension $n \geq 2$ is always assumed to be isolated, strictly log canonical of index one, where a strictly log canonical singularity means a log canonical and not log terminal singularity. Let $f : Y \longrightarrow X$ be a good resolution (*i.e.*, a resolution with the simple normal crossing exceptional divisor) of the singularity of X. We have

$$K_Y = f^*K_X - E + F,$$

where $E = E_{red} > 0$ and $F \geq 0$ have no common components. The divisor E is called the essential part of the exceptional divisor on the resolution. For a simple normal crossing divisor E, we associate a simplicial complex Γ_E called the dual complex in a canonical way. Fujino defines an invariant $\mu(X, x)$ and it turns out to be

$$\mu = \mu(X, x) = \min\{\dim W \mid W \text{is a stratum of } E\}$$

(see [1, 4.11]). Note that $\dim \Gamma_E = n - \mu - 1$.

On the other hand, we define the Hodge type of the singularity (X, x) in the following way: Since

$$\mathbb{C} = H^{n-1}(E, \mathcal{O}_E) \simeq Gr_F^0 H^{n-1}(E, \mathbb{C}) \simeq \bigoplus_{i=0}^{n-1} H^{0,i}_{n-1}(E),$$

there is unique i such that $H^{0,i}_{n-1}(E) \neq 0$, where $H^{0,i}_{n-1}(E)$ is $(0, i)$-Hodge component of $H^{n-1}(E, \mathbb{C})$ and F is the Hodge filtration. In this case, we call the singularity (X, x) of type $(0, i)$. We can easily prove that the type is independent of the choice of resolutions ([2]).

One of the main results (Theorem 5.5) in [1] states that for (X, x) of type $(0, i)$, the equality $\mu(X, x) = i$ holds. Theorem 1 below states the same conclusion and its proof was privately communicated by the author to Fujino in 1999 (cited as [I3] in the reference list of [1]). The author thinks that it is reasonable to publish the original proof as a supplement to Fujino's article, because her original proof is simpler and used only classical method, while Fujino uses recent results in minimal model theory.

Theorem 1. *Let E be the essential part of the exceptional divisor of a good resolution $Y \longrightarrow X$ of an n-dimensional isolated strictly log canonical singularity (X, x).If the Hodge type is of $(0, i)$, then* $\dim \Gamma_E = n - i - 1$.

The following lemma appeared in [3, Lemma 7.4.9]. As it is written in Japanese, we write the proof down here for the non-Japanese readers.

Lemma 2. *Let E be a simple normal crossing divisor on an n-dimensional non-singular variety. If $H^{0,i}_{n-1}(E) \neq 0$, then* $\dim \Gamma_E \geq n - i - 1$.

Proof. After renumbering the suffixes if necessary, we prove that there exist $n - i$ irreducible components $E_1, \dots, E_{n-i}$ such that $E_1 \cap \cdots \cap E_{n-i} \neq \emptyset$. Let E' be a minimal subdivisor of E such that $H^{0,i}_{n-1}(E') \neq 0$. If E' is irreducible, then it is a non-singular variety of dimension $n-1$, therefore we obtain $i = n - 1$ by the basic fact in mixed Hodge theory (see for example [3, Theorem 7.1.6]). Therefore,

$$\dim \Gamma_E \geq 0 = n - (n - 1) - 1,$$

i.e., the required inequality becomes trivial. If E' is not irreducible, take an irreducible component $E_1 < E'$ and decompose E' as $E' = E_1 + E_1^\vee$. Then by the minimality of E', we have $H^{0,i}_{n-1}(E_1) = H^{0,i}_{n-1}(E_1^\vee) = 0$. Consider the exact sequence:

$$H^{n-2}(E_1 \cap E_1^\vee, \mathbb{C}) \longrightarrow H^{n-1}(E', \mathbb{C}) \longrightarrow H^{n-1}(E_1, \mathbb{C}) \oplus H^{n-1}(E_1^\vee, \mathbb{C}).$$

By the above vanishing, the $(0, i)$-component of the center term comes from the left term, therefore $i \leq n - 2$ and $H^{0,i}_{n-2}(E_1 \cap E_1^\vee) \neq 0$.

Take $E_1^\dagger$, a minimal subdivisor of $E_1^\vee$ such that $H^{0,i}_{n-2}(E_1 \cap E_1^\dagger) \neq 0$. If $E_1 \cap E_1^\dagger$ is irreducible, then it is a non-singular variety of dimension $n - 2$, therefore we obtain $i = n - 2$ by the basic fact in mixed Hodge theory. Therefore,

$$\dim \Gamma_E \geq \dim \Gamma_{E_1 + E_1^\dagger} \geq 1 = n - (n - 2) - 1,$$

i.e., the required inequlatiy holds. If $E^\dagger = E_1 \cap E_1^\dagger$ is not irreducible, take an irreducible component E_2 of $E^\dagger$ such that the decomposition

$E^\dagger = E_2 + E_2^\vee$ gives a non-trivial decomposition $E_1 \cap E_1^\dagger = E_1 \cap E_2 + E_1 \cap E_2^\vee$. By the same argument as above, we obtain $i \le n-3$ and $H^{0,i}_{n-3}(E_1 \cap E_2 \cap E_2^\vee) \neq 0$. Continue this procedure successively until we eventually obtain

$$H^{0,i}_i(E_1 \cap E_2 \cap \cdots \cap E_{n-i-1} \cap E^\vee_{n-i-1}) \neq 0,$$

which yields $E_1 \cap E_2 \cap \cdots \cap E_{n-i-1} \cap E^\vee_{n-i-1} \neq \emptyset$. □

Proof of Theorem 1. The inequality $\geq$ is proved in Lemma 2. Assume the strict inequality. Then there exist components $E_1, \ldots\ E_s$, $(s > n-i)$ such that $C := E_1 \cap \ldots \cap E_s \neq \emptyset$. We may assume that $E_j \cap C = \emptyset$ for any E_j $(j > s)$. Let $\varphi : Y' \longrightarrow Y$ be the blow-up at C, E' the reduced total pull-back of E, E_0 the exceptional divisor for φ and E'_j the proper transform of E_j. Then E' is again the essential part on Y' and E' itself is a minimal subdivisor of E' such that $H^{0,i}_{n-1}(E') \neq 0$ by [2, Corollary 3.9]. Make the procedure of the proof of the lemma with taking E_0 as E_1 in the lemma. Then we obtain $E'_1, \ldots, E'_{n-i-1}$ (by renumbering the suffices $1, \ldots, s$) such that $H^{0,i}_i(E_0 \cap E'_1 \cap \ldots \cap E'_{n-i-1}) \neq 0$. On the other hand, i-dimensional variety $E_0 \cap E'_1 \cap \ldots \cap E'_{n-i-1}$ is a $\mathbb{P}^{s-n+i}$-bundle over C, because it is the exceptional divisor of the blow up of an $(i+1)$-dimensional variety $E_1 \cap \cdots \cap E_{n-i-1}$ with the $(n-s)$-dimensional center C. By the assumption on s, we note that $s - n + i > 0$. Hence we have $H^i(E_0 \cap E'_1 \cap \ldots \cap E'_{n-i-1}, \mathcal{O}) = 0$. In particular

$$H^{0,i}_i(E_0 \cap E'_1 \cap \ldots \cap E'_{n-i-1}) = 0,$$

a contradiction. □

References

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